International Journal of Theoretical Physics, Vol. 32, No. 6, 1993

Countable Jauch–Piron Logics

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Received April 12, 1992

It is shown that there exist non-Boolean unital and countable quantum logics which are Jauch–Piron.

1. INTRODUCTION

The cardinality of Jauch–Piron logics appears to be a very important aspect of the doctrine of the Jauch–Piron property (Jauch, 1968; Piron, 1976).

The finiteness conditions ensuring that a Jauch–Piron logic is a Boolean algebra were examined by Rüttimann (1977), Bunce *et al.* (1985), and Rogalewicz (1991). The definitive result was obtained by Rogalewicz, who showed that the number of blocks of a unital Jauch–Piron orthomodular poset is finite if and only if it equals 1.

On the other hand, Pták and Pulmannová (1989, 1991) conjectured that there are non-Boolean unital and countable Jauch–Piron logics. They supposed that the corresponding examples might be fairly complicated.

A celebrated theorem of Gleason (1957) yields almost appropriate examples, but unfortunately, uncountable ones.

In the present paper we develop a technique of countable models of orthomodular lattices to prove the existence of non-Boolean countable subortholattices of the orthomodular lattice of all subspaces of $\mathbb{R}^n$, $n \geq 3$, inheriting the Jauch–Piron property. Of course, such logics are necessarily unital.

2. COUNTABLE MODELS OF ORTHOMODULAR LATTICES

Recall that an orthomodular poset (OMP) (e.g., Pták and Pulmannová, 1991) is a poset E with the greatest element 1 and an involutive antiautomorphism $'$: $E \to E$ satisfying the following conditions:

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0020-7748/93/0600-0885$07.00/0

(i) $x \in E \Rightarrow x + x' = 1$.
(ii) $x, y \in E, x \leqslant y \Rightarrow \exists z \in E(y = x + z)$.
(iii) $x, y \in E, x \perp y \Rightarrow x + y$ exists.

(Here for $x, y \in E$ we write $x \perp y$ provided $x \leq y'$ and denote by $x + y$ the supremum of x and y in the case $x \perp y$.)

Obviously $\mathbf{0} \overset{\mathrm{df}}{=} 1'$ is the smallest element of an OMP.

An orthomodular lattice (OML) (e.g., Kalmbach, 1983) is an OMP that is a lattice.

Let E be an OMP. A function $f\colon E \to \mathbb{R}$ is referred to as a finitely additive one provided

$$x, y \in E, \quad x \perp y \ \Rightarrow\ f(x+y) = f(x) + f(y)$$

Let $F(E)$ denote the set of all finitely additive functions $f\colon E \to \mathbb{R}$ that are bounded (i.e., satisfy the condition $\sup_{x \in E} |f(x)| < +\infty$). Then $F(E)$ is obviously a vector space over $\mathbb{R}$ with respect to pointwise addition and multiplication with scalars. [Moreover, $F(E)$ is a Banach space with respect to the norm $\|f\| = \sup_{x \in E} |f(x)|$ $(f \in F(E))$.]

A finitely additive state on E is a finitely additive function $f\colon E \to \mathbb{R}$ such that $f(E) \subset [0, +\infty)$ and $f(1) = 1$. Let $S(E)$ denote the set of all finitely additive states on E. Obviously $S(E)$ is a convex subset of $F(E)$.

Let L be an OML. A subset $L_0 \subset L$ is said to be a subortholattice of L if L_0 is a sublattice of L, $1 \in L_0$ and $x \in L_0 \Rightarrow x' \in L_0$.

Definition 2.1. A model of L is a subortholattice L_0 of L such that any $f \in S(L_0)$ can be uniquely extended to an element of $S(L)$.

Now we state the main result of this section.

Theorem 2.2. If $F(L)$ is finite-dimensional, then there exists a countable model of L.

We start with two simple remarks of a very general character.

Let Ω be a set and V be a finite-dimensional subspace of the vector space $\mathbb{R}^\Omega$ of all functions $f\colon \Omega \to \mathbb{R}$ (throughout, operations on functions as vectors are supposed to be pointwise) and suppose $\dim V = m$. Let us endow Ω with the topology induced by V (i.e., the weakest topology in Ω with respect to which all functions $f \in V$ are continuous).

Remark 2.3. There exists an m-element set $A \subset \Omega$ such that any function $f\colon A \to \mathbb{R}$ can be uniquely extended to a function from V.

Proof. By induction, for each $k \in \{0, \ldots, m\}$ there exists $A_k \subset \Omega$ such that $\operatorname{card} A_k = k$ and any function $f\colon A_k \to \mathbb{R}$ can be extended to a function from V. Put $A = A_m$. QED

Remark 2.4. The topological space Ω has a countable base. In particular, Ω is separable.

Proof. Take basic vectors $e_1, \ldots, e_m \in V$. Consider finite intersections of sets that are of the form $e_k^{-1}((a, b))$, $a, b \in \mathbb{R}$ being rational. QED

Next, suppose $\dim F(L) = m$. We endow L with the topology induced by $F(L)$. By Remark 2.3, there exists an m-element $A \subset L$ such that any function $f\colon A \to \mathbb{R}$ can be uniquely extended to a function from $F(L)$. By Remark 2.4, there exists a countable and dense $B \subset L$ satisfying $A \subset B$. Put

$$U = \{f\colon L \to [-1, 1] \mid A \cup \{0\} \subset f^{-1}(0)\}$$

and

$$\Delta = \Gamma_0 \cup \Gamma$$

where

$$\Gamma_0 = \{\{x, x'\} \mid x \in L \backslash \{0, 1\}\}$$

$$\Gamma = \{\{x, y, z\} \subset L \backslash \{0\} \mid x + y + z = 1\}$$

Remark 2.5. Let $f\colon L \to \mathbb{R}$. It is easy to verify that f is finitely additive if and only if $f(0) = 0$ and $\alpha \in \Delta \Rightarrow \sum_{x \in \alpha} f(x) = f(1)$. (The assertion as well as the following lemma can be extended to arbitrary OMPs.)

Lemma 2.6. For every $x \in L$ and $\varepsilon > 0$ there exists a finite set $\Delta_{x,\varepsilon} \subset \Delta$ satisfying

$$f \in U, \quad \forall \alpha \in \Delta_{x,\varepsilon} \left(\sum_{y \in \alpha} f(y) = f(1) \right) \Rightarrow |f(x)| < \varepsilon$$

Proof. For each $\alpha \in \Delta$ put

$$\alpha^* = \left\{ f \in U \,\middle|\, \sum_{y \in \alpha} f(y) = f(1) \text{ and } |f(x)| \geq \varepsilon \right\}$$

Suppose $f \in \bigcap_{\alpha \in \Delta} \alpha^*$. By Remark 2.5, we obtain $f \in F(L)$. Obviously $A \subset f^{-1}(0)$ and $f(x) \neq 0$. This is a contradiction. Thus $\bigcap_{\alpha \in \Delta} \alpha^* = \varnothing$. Sets of the form α^* ($\alpha \in \Delta$) are closed in the product topological space $[-1, 1]^L$, which is compact by the Tikhonoff theorem. Therefore, there exists a finite $\Delta_{x,\varepsilon} \subset \Delta$ with $\bigcap_{\alpha \in \Delta_{x,\varepsilon}} \alpha^* = \varnothing$. QED

Proof of the Theorem. For arbitrary $x \in L$ and $\varepsilon > 0$ put

$$\delta_{x,\varepsilon} = \bigcup_{\alpha \in \Delta_{x,\varepsilon}} \alpha$$

For every $T \subset L$ let $\tilde{T}$ denote the smallest, with respect to inclusion, sub-ortholattice of L containing T. (Note that if T is countable, then $\tilde{T}$ is countable.) Now, put $L_1 = \tilde{B}$,

$$L_{k+1} = \widetilde{L_k \cup \bigcup_{x \in L_k} \delta_{x,1/k}} \qquad (k \in \mathbb{N})$$

and $L_0 = \bigcup_{k=1}^{\infty} L_k$. Let us show that L_0 is a countable model of L.

It is clear that L_0 is a countable subortholattice of L, L_0 is dense in L, and $A \subset L_0$. Suppose that $f \in S(L_0)$. Then there exists $g \in F(L)$ satisfying $g|_A = f|_A$. Put $h = f - g|_{L_0}$. Then $h \in F(L_0)$. Take $\lambda \in \mathbb{R} \backslash \{0\}$ such that $|\lambda h(x)| \leq 1$ for all $x \in L_0$. Put

$$\varphi(x) = \begin{cases} \lambda h(x) & \text{if} \quad x \in L_0 \\ 0 & \text{if} \quad x \in L \backslash L_0 \end{cases}$$

Obviously $\varphi \in U$. Let $x \in L_0$. Since $L_1 \subset L_2 \subset \cdots \subset L_k \subset \cdots$, it follows that there exists $k_0 \in \mathbb{N}$ satisfying $k \geq k_0 \Rightarrow x \in L_k$. Since $\delta_{x,1/k} \subset L_0$ $(k \geq k_0)$, we get $\sum_{y \in \alpha} \varphi(y) = \varphi(1)$ for all $k \geq k_0$ and $\alpha \in \Delta_{x,1/k}$. Hence $|\varphi(x)| < 1/k$ $(k \geq k_0)$. Thus $\varphi(x) = 0$ and therefore $f = g|_{L_0}$. Since L_0 is dense in L and $g(x) \geq 0$ $(x \in L_0)$ for a continuous function g, it follows that $g \in S(L)$. The uniqueness is obvious. QED

3. JAUCH–PIRON PROPERTY

A quantum logic (QL) (Pták and Pulmannová, 1991) is a σ-orthocomplete OMP, i.e., an OMP E such that for any sequence (x_i) in E satisfying $x_i \perp x_j$ $(i \neq j)$ there exists the supremum $\bigvee_{i=1}^{\infty} x_i$ (usually written as $\sum_{i=1}^{\infty} x_i$).

Let E be a QL. A state on E is a mapping $f: E \to \mathbb{R}$ such that $f(E) \subset [0, +\infty)$, $f(1) = 1$, and $f(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{\infty} f(x_i)$ for every sequence (x_i) in E with the property $x_i \perp x_j$ $(i \neq j)$.

Obviously if every chain in E is finite, then the concepts of a state and a finitely additive state on E coincide.

Let f be a state on E. Then f is said to be Jauch–Piron providing

$$x, y \in E, \quad f(x) = f(y) = 1 \Rightarrow \exists z \in E \quad [z \leq x, z \leq y, f(z) = 1]$$

A QL E is called Jauch–Piron if every state on E is Jauch–Piron.

For each $n \in \mathbb{N}$ let $L(\mathbb{R}^n)$ denote the orthomodular lattice of all subspaces of $\mathbb{R}^n$. Observe that $L(\mathbb{R}^n)$ is a QL whose every chain is finite.

Proposition 3.1. If $n \geq 3$, then there exists a countable Jauch–Piron subortholattice of $L(\mathbb{R}^n)$ that is not a Boolean algebra.

Proof. By the theorem of Gleason (1957), $F(L(\mathbb{R}^n))$ is finite-dimensional. By Theorem 2.2, there exists a countable model L_0 of $L(\mathbb{R}^n)$. Since by the same theorem of Gleason, $L(\mathbb{R}^n)$ is Jauch–Piron, we simply obtain that L_0 is Jauch–Piron, too. Clearly any Boolean subalgebra of $L(\mathbb{R}^n)$ is finite and thus its set of all states has only a finite set of extreme points. The set of all extreme points of $S(L(\mathbb{R}^n))$ and hence of $S(L_0)$ is obviously infinite. Thus L_0 is not a Boolean algebra. QED

The following corollary yields the affirmative answer to the question of Pták and Pulmannová (1991).

Recall that a QL E is called unital if for any $x \in E \backslash \{0\}$ there exists a state f on E with $f(x)=1$.

Corollary 3.2. There exist unital and countable QLs which are Jauch–Piron and are not Boolean algebras.

4. APPENDIX. AXIOMS FOR Γ_0 AND Γ

Let E be an OMP with $\mathbf{0} \neq 1$. Put (as in the text)

$$\Gamma_0 = \{\{x, x'\} \mid x \in E \backslash \{0, 1\}\}$$

$$\Gamma = \{\{x, y, z\} \subset E \backslash \{0\} \mid x + y + z = 1\}$$

We also put $\mathscr{D} = E \backslash \{0, 1\}$.

Clearly Γ_0 is a partition of $\mathscr{D}$ consisting of two-element sets. We can identify such partitions with mappings ${}^\theta\colon \mathscr{D} \to \mathscr{D}$ satisfying $x \in \mathscr{D} \Rightarrow x^{\theta\theta} = x \neq x^\theta$ (one should take $\Gamma_0 = \{\{x, x^\theta\} \mid x \in \mathscr{D}\}$).

Obviously we can restore the OMP E with the help of $\mathscr{D}$, Γ_0, and Γ:

$$E = \mathscr{D} \cup \{0, 1\}, \quad \mathbf{0}' = 1, \quad 1' = \mathbf{0}, \quad x' = x^\theta \quad (x \in \mathscr{D})$$

and if $x, y \in E$, then $x \leq y \Leftrightarrow x = y$ or $x = \mathbf{0}$ or $y = 1$ or $x, y \in \mathscr{D}$ and $\exists z \in \mathscr{D}(\{x, y^\theta, z\} \in \Gamma)$.

So, the question of axioms for Γ_0 and Γ arises.

Let $\mathscr{D}$ be a set, let ${}^\theta\colon \mathscr{D} \to \mathscr{D}$, satisfy

$$x \in \mathscr{D} \;\Rightarrow\; x^{\theta\theta} = x \neq x^\theta$$

and let Γ be a set of three-element subsets of $\mathscr{D}$.

The following statement gives another characterization of OMPs.

Theorem 4.1. (Ovchinnikov, 1983). The following two conditions are equivalent:

1. There exists an OMP E with $\mathbf{0} \neq 1$, $\mathscr{D} = E \backslash \{0, 1\}$, ${}^\theta = {}'|_{\mathscr{D}}$, and

$$\Gamma = \{\{x, y, z\} \subset E \backslash \{0\} \mid x + y + z = 1\}$$

2. (G_1)

$$\{x, a, b\}, \{x^\theta, c, d\} \in \Gamma \;\Rightarrow\; \exists y \in \mathscr{D} \;(\{a, c, y\} \in \Gamma)$$

and (G_2)

$$\{x_1, x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_5, x_6, x_1\} \in \Gamma, \; x_1 \neq x_4 \;\Rightarrow$$

$$\exists y \in \mathscr{D} \;(\{x_1, x_4^\theta, y\} \in \Gamma)$$

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